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Completeness, Compactness, and Precompactness in Fuzzy Uniform Spaces: Part I

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INTRODUCTION

It is a well-known fact that the compactness of a uniform space breaks down into completeness and precompactness of that space. It is therefore a natural question to ask whether for fuzzy uniform spaces significant notions of completeness and precompactness can be introduced which will also lead to a decomposition of compactness.

To solve this problem, we had to introduce some notion of Cauchy prefilters. In doing so we discovered the remarkable fact that there exist two extensions of the usual notion of Cauchy filters which live together in harmony, each of which has a particular role to play and one being useless or even meaningless without the other. The main characterization of one type is that they are prefilters containing *small* fuzzy sets, while the main characterization of the other type is that they contain the class of convergent prefilters.

We prove that the notions of precompactness and of completeness which we then introduce are good extensions. Finally we prove (a) that a compact fuzzy uniform space has as fuzzy entourages exactly the fuzzy neighborhoods of the diagonal; (b) that a continuous function on a compact fuzzy uniform space is uniformly continuous; and finally, of course, (c) that in fuzzy uniform spaces compactness is equivalent to precompactness plus completeness.

We would like to mention that we can construct a completion of a fuzzy uniform space which has nice properties. Since this, however, would lead us beyond the scope of the present article, we shall develop this construction in the second part.

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1. Preliminaries

As usual, the unit interval is denoted I, while $I_0 =]0, 1]$ and $I_1 = [0, 1[$. Filters are denoted by capital script letters, prefilters by Fraktur letters, and fuzzy sets or values in I by lowercase Greek letters. For definitions and results on prefilters and on convergence we refer the reader to [4]. We do, however, recall that if $\mathfrak{F}$ and $\mathfrak{G}$ are prefilters such that $\mu \wedge \nu \neq 0$ for all $\mu \in \mathfrak{F}$ and $\nu \in \mathfrak{G}$, then $\mathfrak{F} \vee \mathfrak{G}$ denotes the prefilter $\{\mu \wedge \nu, \mu \in \mathfrak{F}, \nu \in \mathfrak{G}\}$. If $\mathfrak{F}$ is a prefilter with characteristic value $c(\mathfrak{F}) = 1$, then we denote by $\iota_1(\mathfrak{F})$ the filter $\{\mu^{-1}]\varepsilon, 1] \mid \mu \in \mathfrak{F}, \varepsilon \in I_1\}$. If $\mathscr{F}$ is a filter, then we denote by $\omega_1(\mathscr{F})$ the prefilter

$$\{\mu \mid \forall \varepsilon \in I_1 : \mu^{-1}]\varepsilon, 1] \in \mathscr{F}\}.$$

The following notion, as opposed to the characteristic value of a prefilter [4], will play an important role:

Definition 1.1. If $\mathfrak{F}$ is a prefilter, then we define its *lower characteristic value* as

$$c^-(\mathfrak{F}) = \inf_{\mathfrak{G} \in \mathscr{F}_m(\mathfrak{F})} c(\mathfrak{G}).$$

Definition 1.2. *If* (X, Δ) is a fuzzy topological space, then we shall say that a prefilter $\mathfrak{F}$ is *convergent* if and only if

$$\sup_{x \in X} \lim \mathfrak{F}(x) = c^-(\mathfrak{F}).$$

Obviously, $\mathfrak{F}$ is convergent if and only if the inequality $\geqslant$ holds.

For definitions and results on fuzzy neighborhood spaces we refer to [5]. There the following operations were introduced: let $\mathfrak{B}$ be a prefilterbasis, then

$$\hat{\mathfrak{B}} = \left\{ \sup_{\varepsilon \in I_0} (\beta_\varepsilon - \varepsilon) \mid (\beta_\varepsilon)_{\varepsilon \in I_0} \in \mathfrak{B}^{I_0} \right\},$$

$$[\mathfrak{B}] = \{\mu \in I^X \mid \exists \beta \in \mathfrak{B} : \beta \leqslant \mu\},$$

and $\tilde{\mathfrak{B}} = [\hat{\mathfrak{B}}] = \widehat{[\mathfrak{B}]}$.

In [5] a generalization of the notion of characteristic value was also introduced; let $\mathfrak{F}$ and $\mathfrak{G}$ be prefilters, then

$$\begin{aligned} c(\mathfrak{F}, \mathfrak{G}) &= c(\mathfrak{F} \vee \mathfrak{G}), &&\text{if } \mathfrak{F} \vee \mathfrak{G} \text{ exists}, \\ &= 0, &&\text{otherwise.} \end{aligned}$$

The next proposition, the proof of which is left to the reader, will be used often.

PROPOSITION 1.1. *The following properties hold for prefilters* $\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{G}$, and $\mathfrak{G}'$:

(i) $\mathfrak{F} \supset \mathfrak{F}'$ *and* $\mathfrak{G} \supset \mathfrak{G}' \Rightarrow c(\mathfrak{F}, \mathfrak{G}) \leqslant c(\mathfrak{F}', \mathfrak{G}')$,

(ii) $\mathfrak{F} \supset \mathfrak{G}$ *or* $\mathfrak{F}' \supset \mathfrak{G} \Rightarrow c(\mathfrak{F}, \mathfrak{F}') \leqslant c(\mathfrak{G})$,

(iii) $\mathfrak{F} \supset \mathfrak{G}$ *and* $\mathfrak{F} \supset \mathfrak{G}' \Rightarrow c(\mathfrak{F}) \leqslant c(\mathfrak{G}, \mathfrak{G}')$,

(iv) $c^-(\mathfrak{F}) \leqslant c(\mathfrak{F})$,

(v) *if* $\mathfrak{F}$ *is prime, then* $c^-(\mathfrak{F}) = c(\mathfrak{F})$.

We recall that a fuzzy neighborhoodspace is a pair $(X, (\mathfrak{V}(x))_{x \in X})$, where $(\mathfrak{V}(x))_{x \in X}$ is a family of prefilters on X fulfilling the conditions:

(N1) For all $x \in X$ and for all $\nu \in \mathfrak{V}(x)$: $\nu(x) = 1$.

(N2) For all $x \in X$: $\widehat{\mathfrak{V}(x)} = \mathfrak{V}(x)$.

(N3) For all $x \in X$, for all $\nu \in \mathfrak{V}(x)$ and for all $\varepsilon \in I_0$ there exists a family $(\nu_z^\varepsilon)_{z \in X}$ such that $\nu_z^\varepsilon \in \mathfrak{V}(z)$ for all $z \in X$ and such that

$$\sup_{z \in X} \nu_x^\varepsilon(z) \wedge \nu_z^\varepsilon(y) - \varepsilon \leqslant \nu(y)$$

for all $y \in X$.

The members of $\mathfrak{V}(x)$ are called fuzzy neighborhoods of x.

DEFINITION 1.3. If $(X, (\mathfrak{V}(x))_{x \in X})$ is a fuzzy neighborhoodspace and $\mathfrak{F}$ is a prefilter on X, then we say that $\mathfrak{F}$ fulfills the *kernel condition* if and only if for all $\mu \in \mathfrak{F}$ and for all $\varepsilon \in I_0$, there exists $\mu' \in \mathfrak{F}$ and $(\nu_z^\varepsilon)_{z \in X}$ such that $\nu_z^\varepsilon \in \mathfrak{V}(z)$ for all $z \in X$ and such that $\sup_{z \in X} \mu'(z) \wedge \nu_z^\varepsilon(y) - \varepsilon \leqslant \mu(y)$ for all $y \in X$.

Condition (N3) then simply says that each $\mathfrak{V}(x)$ fulfills the kernel condition.

The fuzzy topology defined by a fuzzy neighborhoodsystem is given by the fuzzy closure operator

$$\bar{\mu}(x) = \inf_{\nu \in \mathfrak{V}(x)} \sup_{y \in X} \nu \wedge \mu(y)$$

for all $\mu \in I^X$ and $x \in X$.

The following is a straightforward extension of the notion of fuzzy neighboorhood of a point:

DEFINITION 1.4. If $(X, (\mathfrak{V}(x))_{x\in X})$ is a fuzzy neighborhoodspace and $Y \subset X$, then we say that ν is a *fuzzy neighborhood* of Y if and only if $\nu \in \mathfrak{V}(y)$ for all $y \in Y$.

For definitions and results on fuzzy uniform spaces we refer to [6]. Let us recall, however, that a fuzzy uniform space is a pair $(X, \mathfrak{U})$, where $\mathfrak{U}$ fulfills the following conditions:

(FU1) $\mathfrak{U}$ is a prefilter on $X \times X$.

(FU2) $\hat{\mathfrak{U}} = \mathfrak{U}$.

(FU3) For all $\nu \in \mathfrak{U}$ and $x \in X$: $\nu(x, x) = 1$.

(FU4) For all $\nu \in \mathfrak{U}$: ${}_s\nu \in \mathfrak{U}$.

(FU5) For all $\nu \in \mathfrak{U}$ and for all $\varepsilon \in I_0$, there exists $\nu_\varepsilon \in \mathfrak{U}$ such that $\nu_\varepsilon \circ \nu_\varepsilon - \varepsilon \leqslant \nu$,

where ${}_s\nu$ is defined by ${}_s\nu(x, y) = \nu(y, x)$ and $\nu \circ \nu'$ by $\nu \circ \nu'(x, y) = \sup_{z\in X} \nu'(x, z) \wedge \nu(z, y)$.

The members of $\mathfrak{U}$ are called *fuzzy entourages*. We denote by ${}_s\mathfrak{U}$ the set of those $\nu \in \mathfrak{U}$ for which ${}_s\nu = \nu$. If $\mathscr{U}$ is a uniformity on X, we denote by $\omega_u(\mathscr{U})$ the fuzzy uniformity

$$\{\mu \in I^{X\times X} \mid \forall \varepsilon \in I_1 : \mu^{-1}\,]\varepsilon, 1] \in \mathscr{U}\}.$$

The fuzzy topology determined by a fuzzy uniformity is given by the fuzzy closure operator $\bar{\mu} = \inf_{\nu\in\mathfrak{U}} \nu\langle\mu\rangle$ for all $\mu \in I^X$, where $\nu\langle\mu\rangle$ is defined by $\nu\langle\mu\rangle(x) = \sup_{y\in X} \mu(y) \wedge \nu(y, x)$.

A characterization of compactness in fuzzy uniform spaces was proved in the more general setting of fuzzy neighborhoodspaces in [7]. We shall, however, need only the version we give now.

THEOREM 1.2. *A fuzzy uniform space $(X, \mathfrak{U})$ is compact if and only if for all family $(\nu_x)_{x\in X} \in \mathfrak{U}^X$ and for all $\varepsilon \in I_0$ there exists a finite subfamily $(\nu_x)_{x\in Y}$ such that* $\sup_{x\in Y} \nu_x\langle x\rangle \geqslant 1 - \varepsilon$.

2. HYPER CAUCHY PREFILTERS AND CAUCHY PREFILTERS

DEFINITION 2.1. A prefilter $\mathfrak{C}$ in a fuzzy uniform space $(X, \mathfrak{U})$ is called a *hyper Cauchy prefilter* if and only if the following conditions are fulfilled:

(HC1) $c(\mathfrak{C}) = 1$.

(HC2) $\hat{\mathfrak{C}} = \mathfrak{C}$, i.e., for all $(\mu_\varepsilon)_{\varepsilon\in I_0} \in \mathfrak{C}^{I_0}$, we have that $\sup_{\varepsilon\in I_0} (\mu_\varepsilon - \varepsilon) \in \mathfrak{C}$.

(HC3) For all $\nu \in \mathfrak{U}$ and for all $\varepsilon \in I_0$ there exists $\mu_\varepsilon \in \mathfrak{C}$ such that $\mu_\varepsilon \times \mu_\varepsilon - \varepsilon \leqslant \nu$.

Remark that (HC3) can also be expressed by saying that for all $\nu \in \mathfrak{U}$ there exists a family $(\mu_\varepsilon)_{\varepsilon \in I_0} \in \mathfrak{C}^{I_0}$ such that $\sup_{\varepsilon \in I_0} (\mu_\varepsilon \times \mu_\varepsilon - \varepsilon) \leqslant \nu$. Such a family will be referred to as a ν-small family.

DEFINITION 2.2. A hyper Cauchy prefilter is called *minimal* if and only if, for the inclusion relation, it is minimal in the set of all hyper Cauchy prefilters, i.e., if and only if there exists no strictly coarser hyper Cauchy prefilter.

DEFINITION 2.3. A prefilterbasis $\mathfrak{B}$ is called a *hyper Cauchy prefilterbasis* if and only if the following conditions are fulfilled:

(BHC1) $c(\mathfrak{B}) = 1$.

(BHC2) For all $\nu \in \mathfrak{U}$ and for all $\varepsilon \in I_0$ there exists $\beta_\varepsilon \in \mathfrak{B}$ such that $\beta_\varepsilon \times \beta_\varepsilon - \varepsilon \leqslant \nu$.

DEFINITION 2.4. If $\mathfrak{C}$ is a hyper Cauchy prefilter, then we say that $\mathfrak{B}$ is a *basis* for $\mathfrak{C}$ if and only if $\mathfrak{B}$ is a prefilterbasis and $\tilde{\mathfrak{B}} = \mathfrak{C}$.

PROPOSITION 2.1. *If $\mathfrak{B}$ is a hyper Cauchy prefilterbasis, then $\tilde{\mathfrak{B}}$ is a hyper Cauchy prefilter with $\mathfrak{B}$ as basis. Conversely, if $\mathfrak{C}$ is a hyper Cauchy prefilter with $\mathfrak{B}$ as basis, then $\mathfrak{B}$ is a hyper Cauchy prefilterbasis.*

Proof. By straightforward verification.

PROPOSITION 2.2. *If $(X, \mathfrak{U})$ is a fuzzy uniform space and $\mathfrak{C}$ is a hyper Cauchy prefilter, then there exists a unique minimal hyper Cauchy prefilter $\mathfrak{C}_0$ coarser than $\mathfrak{C}$. Moreover, a basis for $\mathfrak{C}_0$ is given by $\mathfrak{B} = \{\nu\langle\mu\rangle : \nu \in \mathfrak{U}, \mu \in \mathfrak{C}\}$, i.e., $\mathfrak{C}_0 = \tilde{\mathfrak{B}}$.*

Proof. That $\mathfrak{B}$ is a prefilterbasis follows at once from the facts that $\mathfrak{B} \subset \mathfrak{C}$ and that for any $\nu, \nu' \in \mathfrak{U}$ and $\mu, \mu' \in \mathfrak{C}$ we have $\nu\langle\mu\rangle \wedge \nu'\langle\mu'\rangle \geqslant \nu \wedge \nu'\langle\mu \wedge \mu'\rangle$. From $\mathfrak{B} \subset \mathfrak{C}$, (BHC1) can be deduced at once. To prove (BHC2), let $\nu \in \mathfrak{U}$ and let $\varepsilon \in I_0$. Choose $\xi \in {}_s\mathfrak{U}$ such that $\xi^3 - \varepsilon/2 \leqslant \nu$, then choose $\mu \in \mathfrak{C}$ such that $\mu \times \mu - \varepsilon/2 \leqslant \xi$. Then it follows that for any $x, y \in X$ we have

$$\begin{aligned}
\xi\langle\mu\rangle \times \xi\langle\mu\rangle(x, y) &= \sup_{t,s} \mu(t) \wedge \xi(t, x) \wedge \mu(s) \wedge \xi(s, y) \\
&\leqslant \sup_{t,s} \xi(x, t) \wedge (\xi(t, s) + \varepsilon/2) \wedge \xi(s, y) \\
&\leqslant \overset{3}{\xi}(x, y) + \frac{\varepsilon}{2} \leqslant \nu(x, y) + \varepsilon.
\end{aligned}$$

To show that $\mathfrak{C}_0$ is minimal, let $\mathfrak{C}_1$ be a hyper Cauchy prefilter coarser than $\mathfrak{C}$. Let $\nu \in \mathfrak{U}$, $\mu \in \mathfrak{C}$, and $\varepsilon \in I_0$. Since $\mathfrak{C}_1$ is hyper Cauchy, there exists $\xi \in \mathfrak{C}_1$ such that $\xi \times \xi - \varepsilon \leqslant \nu$. Then

$$\begin{aligned}
\nu\langle\mu\rangle(x) &= \sup_{y \in X} \mu(y) \wedge \nu(y, x) \\
&\geqslant \sup_{y \in X} \mu(y) \wedge (\xi \times \xi(y, x) - \varepsilon) \\
&\geqslant \left(\sup_{y \in X} \mu \wedge \xi(y)\right) \wedge \xi(x) - \varepsilon \\
&= \xi(x) - \varepsilon.
\end{aligned}$$

Consequently, $\nu\langle\mu\rangle \in \mathfrak{C}_1$ which proves that $\mathfrak{C}_0 \subset \mathfrak{C}_1$ and therefore that $\mathfrak{C}_0$ is unique.

In the sequel, if $\mathfrak{C}$ is a hyper Cauchy prefilter, then the minimal hyper Cauchy prefilter coarser than $\mathfrak{C}$ shall consistently be denoted $\mathfrak{C}_0$. Analogously, if we are working in a uniform space and $\mathscr{G}$ is a Cauchy filter, the unique minimal Cauchy filter coarser than $\mathscr{G}$ shall be denoted $\mathscr{G}_0$.

PROPOSITION 2.3. *If $(X, \mathfrak{U})$ is a fuzzy uniform space, then for each $x \in X$ the fuzzy neighborhood prefilter $\mathfrak{U}(x) = \{\nu\langle x\rangle : \nu \in \mathfrak{U}\}$ is a minimal hyper Cauchy prefilter.*

Proof. This is an immediate consequence of Proposition 2.2 if one remarks that for each $x \in X$ the prefilter $\dot{1}_x$ is hyper Cauchy and $\mathfrak{U}(x) = (\dot{1}_x)_0$.

PROPOSITION 2.4. *If $(X, \mathfrak{U})$ is a fuzzy uniform space and $\mathfrak{C}$ is a minimal hyper Cauchy prefilter, then $\mathfrak{C}$ fulfills the kernel condition.*

Proof. This follows at once from Proposition 2.2 since the minimality of $\mathfrak{C}$ implies that for all $\mu \in \mathfrak{C}$ and $\varepsilon \in I_0$ there exists $\mu' \in \mathfrak{C}$ and $\nu \in \mathfrak{U}$ such that $\nu\langle\mu'\rangle - \varepsilon \leqslant \mu$.

In the foregoing the reader will not fail to notice the similarity between our results and methods of proof and their classical analogues. Much of this similarity, however, ends here. Cauchy filters and minimal Cauchy filters exhibit two important aspects of the classical theory. The former provide us with the natural generalization of convergent filters in a uniform space and the latter provide us with the underlying set of the completion of a uniform space. Hyper Cauchy prefilters cannot play the role of Cauchy filters for the simple reason that their characteristic value is always equal to one, whereas that of a convergent prefilter can be any number in I_0. It is thus impossible to prove that a convergent prefilter is hyper Cauchy. Minimal hyper Cauchy prefilters, as we shall show in part two, where we construct a completion of a fuzzy uniform space, can play the role of minimal Cauchy filters mentioned earlier. They also play a key part, however, in defining the right generalization of Cauchy filters.

In what follows, we shall denote by $\mathscr{M}(X)$ the family of all minimal hyper Cauchy prefilters on X.

DEFINITION 2.5. A prefilter $\mathfrak{F}$ on a fuzzy uniform space $(X, \mathfrak{U})$ is called a *Cauchy prefilter* if and only if

$$\sup_{\mathfrak{C} \in \mathscr{M}(X)} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} c(\mathfrak{C}, \mathfrak{G}) = c^-(\mathfrak{F}).$$

From Proposition 1.1 it again follows that for this equality to be fulfilled it is necessary and sufficient that the inequality $\geqslant$ be fulfilled. From Propositions 1.1 and 2.2 it follows that the first member always equals

$$\sup_{\mathfrak{C} \in \mathscr{H}(X)} \inf_{\mathfrak{C} \in \mathscr{P}_m(\mathfrak{F})} c(\mathfrak{C}, \mathfrak{G}),$$

where $\mathscr{H}(X)$ stands for the set of all hyper Cauchy prefilters on X.

THEOREM 2.5. *If $(X, \mathfrak{U})$ is a fuzzy uniform space and $\mathfrak{F}$ is a convergent prefilter, then $\mathfrak{F}$ is a Cauchy prefilter.*

Proof. From Proposition 2.3 it follows that $\mathfrak{U}(x) \in \mathscr{M}(X)$ for all $x \in X$, and from [5, Theorem 7.1] it follows that $\lim \mathfrak{G}(x) = c(\mathfrak{U}(x), \mathfrak{G})$ for all $\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})$. Consequently,

$$\sup_{\mathfrak{C} \in \mathscr{M}(X)} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} c(\mathfrak{C}, \mathfrak{G}) \geqslant \sup_{x \in X} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} \lim \mathfrak{G}(x) = c^-(\mathfrak{F}),$$

which proves the theorem.

Before proceeding to the introduction of the notions of completeness and precompactness, we investigate the relation between hyper Cauchy prefilters and Cauchy prefilters in a little more detail.

Remark that in any fuzzy uniform space, all prefilters defined by

$$\mathfrak{C}(\alpha, x_0) = \{\mu \in I^X \mid \mu(x_0) \geqslant \alpha\} \qquad (x_0 \in X \text{ and } \alpha \in I_0 \text{ fixed})$$

or

$$\mathfrak{C}'(\alpha, x_0) = \{\mu \in I^X \mid \mu(x_0) > \alpha\} \qquad (x_0 \in X \text{ and } \alpha \in I_1 \text{ fixed})$$

are Cauchy prefilters. Among these, only the $\mathfrak{C}(1, x_0) = \dot{1}_{x_0}$ are at the same time hyper Cauchy.

PROPOSITION 2.6. *If $(X, \mathfrak{U})$ is a fuzzy uniform space and $\mathfrak{F}$ is a prefilter which fulfills the following conditions*:

(1) *$\mathfrak{F}$ is prime,*

(2) $\hat{\mathfrak{F}} = \mathfrak{F}$,

(3) $c(\mathfrak{F}) = 1$,

(4) *$\mathfrak{F}$ is Cauchy,*

then $\mathfrak{F}$ is a hyper Cauchy prefilter.

Proof. Let $\nu \in \mathfrak{U}$ and $\varepsilon \in I_0$. Since $\mathfrak{F}$ is Cauchy, we can find $\mathfrak{C} \in \mathscr{M}(X)$ such that $c(\mathfrak{C}, \mathfrak{F}) \geqslant 1 - \varepsilon/2$. Now choose $\xi \in \mathfrak{C}$ such that $\xi \times \xi - \varepsilon \leqslant \nu$ and put $K = \xi^{-1}]1 - \varepsilon, 1]$. If $1_K \notin \mathfrak{F}$, then it follows since $\mathfrak{F}$ is prime that $1_{K^c} \wedge \xi \in \mathfrak{F} \vee \mathfrak{C}$. This, however, is in contradiction with the fact that $c(\mathfrak{C}, \mathfrak{F}) \geqslant 1 - \varepsilon/2$, since $\sup_{x \in X} 1_{K^c} \wedge \xi(x) \leqslant 1 - \varepsilon$. Consequently, $1_K \in \mathfrak{F}$. Then it is easily checked that $1_K \leqslant \xi + \varepsilon$, which implies that $1_K \times 1_K - \varepsilon \leqslant \nu$. This proves that $\mathfrak{F}$ is hyper Cauchy.

PROPOSITION 2.7. *If $(X, \mathfrak{U})$ is a fuzzy uniform space and $\mathfrak{C}$ is a hyper Cauchy prefilter, then $\mathfrak{C}$ is a Cauchy prefilter.*

Proof. Let $\mathfrak{C}$ be hyper Cauchy. Then

$$\sup_{\mathfrak{C}' \in \mathscr{M}(X)} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{C})} c(\mathfrak{G}, \mathfrak{C}') \geqslant \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{C})} c(\mathfrak{G}, \mathfrak{C})$$

$$= \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{C})} c(\mathfrak{G}) = c^{-}(\mathfrak{C}).$$

3. PRECOMPACTNESS AND COMPLETENESS

DEFINITION 3.1. A fuzzy uniform space is called *precompact* if and only if for all $\nu \in \mathfrak{U}$ and for all $\varepsilon \in I_0$ there exists a finite number of points $x_1, \ldots, x_n \in X$ such that $\sup_{i=1}^n \nu\langle x_i \rangle \geqslant 1 - \varepsilon$.

THEOREM 3.1. *Let $(X, \mathfrak{U})$ be a fuzzy uniform space. Then the following properties are equivalent*:

(i) *$(X, \mathfrak{U})$ is precompact.*

(ii) *For all $\nu \in \mathfrak{U}$ and for all $\varepsilon \in I_0$ there exists a finite number of* *fuzzy sets* $\mu_1,\ldots,\mu_n \in I^X$ *such that* $\mu_i \times \mu_i - \varepsilon \leqslant \nu$ *for all* $i = 1,\ldots,n$ *and such that* $\sup_{i=1}^n \mu_i \geqslant 1-\varepsilon$;

(iii) *Each prime prefilter* $\mathfrak{F}$ *such that* $c(\mathfrak{F}) = 1$ *and* $\hat{\mathfrak{F}} = \mathfrak{F}$ *is hyper Cauchy*;

(iv) *Each prime prefilter is Cauchy.*

Proof. (i) $\Rightarrow$ (ii) Let $\nu \in \mathfrak{U}$ and $\varepsilon \in I_0$. Choose $\xi \in {}_s\mathfrak{U}$ such that $\xi \circ \xi - \varepsilon \leqslant \nu$. From the precompactness of $(X, \mathfrak{U})$ it follows that there exist $x_1,\ldots,x_n \in X$ such that $\sup_{i=1}^n \xi\langle x_i\rangle \geqslant 1-\varepsilon$. Clearly, $\{\mu_i = \xi\langle x_i\rangle,\ i = 1,\ldots,n\}$ fulfills the condition of (ii).

(ii) $\Rightarrow$ (iii) Let $\mathfrak{F}$ be prime such that $c(\mathfrak{F}) = 1$ and $\hat{\mathfrak{F}} = \mathfrak{F}$. Also let $\nu \in \mathfrak{U}$ and $\varepsilon \in I_0$. From (ii) we can find $\mu_1,\ldots,\mu_n \in I^X$ such that $\mu_i \times \mu_i - \varepsilon/2 \leqslant \nu$ for all $i = 1,\ldots,n$ and $\sup_{i=1}^n \mu_i \geqslant 1 - \varepsilon/2$. Since $\mathfrak{F}$ is prime there exists $j \in \{1,\ldots,n\}$ such that $(\mu_j + \varepsilon/2) \wedge 1 \in \mathfrak{F}$. Then for any $x, y \in X$

$$((\mu_j + \varepsilon/2) \wedge 1) \times ((\mu_j + \varepsilon/2) \wedge 1)(x,y) = (\mu_j(x) \wedge \mu_j(y) + \varepsilon/2) \wedge 1$$
$$\leqslant \nu(x,y) + \varepsilon,$$

which proves $\mathfrak{F}$ is hyper Cauchy.

(iii) $\Rightarrow$ (iv) Let $\mathfrak{F}$ be a prime prefilter and put

$$\mathfrak{G} = \{\mu \in I^X \mid \forall \varepsilon \in I_1, \exists \nu \in \mathfrak{F} : \mu^{-1}]\varepsilon, 1] \supset \nu^{-1}]0, 1]\}.$$

It is easily verified that $\mathfrak{G}$ is a prime prefilter for which obviously $c(\mathfrak{G}) = 1$ and $\hat{\mathfrak{G}} = \mathfrak{G}$. From (iii) it follows that $\mathfrak{G}$ is a hyper Cauchy prefilter. Now let $\lambda \in \mathfrak{G}$ and $\varepsilon \in I_0$. Then, since $1 - \varepsilon \in I_1$, we can find $\xi_\lambda \in \mathfrak{F}$ such that $\lambda^{-1}]1-\varepsilon, 1] \supset \xi_\lambda^{-1}]0, 1]$. It follows that for any $\nu \in \mathfrak{G}$ and any $\mu \in \mathfrak{F}$

$$\sup_{x \in X} \mu \wedge \lambda(x) + \varepsilon \geqslant \sup_{x \in X} \mu(x) \wedge (\lambda(x) + \varepsilon)$$

$$\geqslant \sup_{x \in X} \mu(x) \wedge \xi_\lambda(x) \wedge (\lambda(x) + \varepsilon)$$

$$\geqslant \sup_{x \in \xi_\lambda^{-1}]0,1]} \mu(x) \wedge \xi_\lambda(x) = \sup_{x \in X} \mu \wedge \xi_\lambda(x) \geqslant c(\mathfrak{F}).$$

Since this holds for all $\varepsilon \in I_0$, it follows that $\mathfrak{F}$ is Cauchy.

(iv) $\Rightarrow$ (i) Suppose $\nu \in \mathfrak{U}$ and $\varepsilon \in I_0$ such that, for all $Y \subset X$ finite, $\sup_{y \in Y} \nu\langle y\rangle \not\geqslant 1 - \varepsilon$. This implies that for all $Y \subset X$ finite

$$F_Y = \{x \mid \sup_{y \in Y} \nu\langle y\rangle(x) < 1 - \varepsilon\} \neq \varnothing.$$

Since $F_Y \cap F_{Y'} = F_{Y \cup Y'}$ for any finite $Y, Y' \subset X$, it follows that

$$\mathscr{F} = [\{F_Y \mid Y \subset X \text{ finite}\}]$$

is a filter on X. Choose an ultrafilter $\mathscr{U} \supset \mathscr{F}$ and put

$$\mathfrak{G} = \{1_U \mid U \in \mathscr{U}\}^{\sim}.$$

Since $\mathfrak{G}$ is clearly prime, $c(\mathfrak{G}) = 1$, and $\hat{\mathfrak{G}} = \mathfrak{G}$, it follows from (iv) that $\mathfrak{G}$ is Cauchy and from Proposition 2.6 that $\mathfrak{G}$ is hyper Cauchy. Thus there exists $U \in \mathscr{U}$ such that $1_U \times 1_U - \varepsilon \leqslant \nu$. If we choose $y_0 \in U$, however, then $F_{\{y_0\}} \in \mathscr{F}$ and consequently $F_{\{y_0\}} \cap U \neq \varnothing$. Choose $x \in F_{\{y_0\}} \cap U$. Then, from $x \in F_{\{y_0\}}$, it follows that $\nu(y_0, x) < 1 - \varepsilon$, while from $x \in U$ it follows that $1 - \varepsilon \leqslant \nu(y_0, x)$. This contradiction proves the last implication.

We shall now prove that Definition 3.1 is a good extension.

THEOREM 3.2. *Let $(X, \mathscr{U})$ be a uniform space. Then $(X, \omega_u(\mathscr{U}))$ is precompact if and only if $(X, \mathscr{U})$ is precompact.*

Proof. Suppose $(X, \mathscr{U})$ is precompact. Let $\nu \in \omega_u(\mathscr{U})$. Then for all $\varepsilon \in I_0$ there exists $U \in \mathscr{U}$ such that $1_U - \varepsilon \leqslant \nu$. Consequently, there exist $F_1, \ldots, F_n \subset X$ such that $\bigcup_{i=1}^n F_i = X$ and such that each F_i is U-small. If we put $\mu_i = (1_{F_i} - \varepsilon) \vee 0$, then it is easily seen that $\sup_{i=1}^n \mu_i \geqslant 1 - \varepsilon$ and that $\mu_i \times \mu_i - \varepsilon \leqslant \nu$ for all $i = 1, \ldots, n$. Conversely, if $(X, \omega_u(\mathscr{U}))$ is precompact, let $U \in \mathscr{U}$. Then $1_U \in \omega_u(\mathscr{U})$. Thus there exist $\mu_1, \ldots, \mu_n \in I^X$ such that $\sup_{i=1}^n \mu_i \geqslant \frac{2}{3}$ and $\mu_i \times \mu_i - \frac{1}{3} \leqslant 1_U$ for all $i = 1, \ldots, n$. It is easily seen that if we put $F_i = \mu_i^{-1}[\frac{2}{3}, 1]$, then $\bigcup_{i=1}^n F_i = X$ and F_i is U-small for each $i = 1, \ldots, n$.

DEFINITION 3.2. A fuzzy uniform space is called *complete* if and only if each Cauchy prefilter is convergent.

Our next aim is to prove that Definition 3.2 is a good extension. To do this we first prove two lemmas which we shall also have to use frequently later on.

LEMMA 3.3. *If $\mathscr{F}$ is a filter on X and $\mathfrak{F}$ is the prefilter defined by*

$$\mathfrak{F} = [\{1_F \mid F \in \mathscr{F}\}],$$

then

$$\hat{\mathfrak{F}} = \omega_1(\mathscr{F}).$$

Proof. It is clear that $\mathfrak{F} \subset \omega_1(\mathscr{F})$. If $(\mu_\varepsilon)_{\varepsilon\in I_0} \in \omega_1(\mathscr{F})^{I_0}$ and $\delta \in I_1$, then

$$\left(\sup_{\varepsilon\in I_0} (\mu_\varepsilon - \varepsilon)^{-1}]\delta, 1]\right) \supset \mu_{\varepsilon_0}^{-1}]\varepsilon_0 + \delta, 1]$$

if $\varepsilon_0 \in I_0$ is chosen such that $\varepsilon_0 + \delta \in I_1$. Since $\mu_{\varepsilon_0}^{-1}]\varepsilon_0 + \delta, 1] \in \mathscr{F}$, it follows that $\omega_1(\mathscr{F})$ satisfies (HC2), and hence $\hat{\mathfrak{F}} \subset \omega_1(\mathscr{F})$.

If, conversely, $\mu \in \omega_1(\mathscr{F})$, $\varepsilon \in I_0$, then $\mu^{-1}]1 - \varepsilon, 1] = F_\varepsilon \in \mathscr{F}$ and $1_{F_\varepsilon} - \varepsilon \leqslant \mu$. Hence

$$\sup_{\varepsilon\in I_0} (1_{F_\varepsilon} - \varepsilon) \leqslant \mu,$$

and therefore $\mu \in \hat{\mathfrak{F}}$.

LEMMA 3.4. (i) *If $(X, \mathfrak{U})$ is a fuzzy uniform space and $\mathfrak{C}$ is a hyper Cauchy prefilter on $(X, \mathfrak{U})$, then $\iota_1(\mathfrak{C})$ is a Cauchy filter on $(X, \iota_u(\mathfrak{U}))$; if, moreover, $\mathfrak{C}$ is minimal, then so is $\iota_1(\mathfrak{C})$.*

(ii) *If $(X, \mathscr{U})$ is a uniform space and $\mathscr{C}$ is a Cauchy filter on $(X, \mathscr{U})$, then $\omega_1(\mathscr{C})$ is a hyper Cauchy prefilter on $(X, \omega_u(\mathscr{U}))$; moreover, $\mathscr{C}$ is minimal if and only if $\omega_1(\mathscr{C})$ is minimal.*

(iii) *If $(X, \mathfrak{U})$ is a fuzzy uniform space and $\mathscr{C}$ is a Cauchy filter on $(X, \iota_u(\mathfrak{U})$, then $\omega_1(\mathscr{C})$ is a hyper Cauchy prefilter on $(X, \mathfrak{U})$; if, moreover, $\omega_1(\mathscr{C})$ is minimal (on $(X, \mathfrak{U})$), then $\mathscr{C}$ is minimal on $(X, \iota_u(\mathfrak{U}))$.*

(iv) *If $(X, \mathscr{U})$ is a uniform space and $\mathfrak{C}$ is a hyper Cauchy prefilter on $(X, \omega_u(\mathscr{U}))$, then $\iota_1(\mathfrak{C})$ is a Cauchy filter on $(X, \mathscr{U})$; moreover, $\mathfrak{C}$ is minimal if and only if $\iota_1(\mathfrak{C})$ is minimal.*

Proof. (i) If $V \in \iota_u(\mathfrak{U})$, then $V = \mu^{-1}]\varepsilon, 1]$ with some $\mu \in \mathfrak{U}$ and some $\varepsilon \in I_1$. Take $\delta > 0$ such that $\varepsilon + \delta < 1$, and then choose $\nu \in \mathfrak{C}$ such that $\nu \times \nu \leqslant \mu + \delta$. Put $A = \nu^{-1}]\varepsilon + \delta, 1] \in \iota_1(\mathfrak{C})$. If $(x, y) \in A \times A$, then $\nu \times \nu(x, y) > \varepsilon + \delta$, and therefore $(x, y) \subset V$, which proves that $A \times A \subset V$. If $\mathfrak{C}$ is minimal, take $A \in \iota_1(\mathfrak{C})$ and thus $A = \mu^{-1}]\delta, 1]$ for some $\delta \in I_1$ and some $\mu \in \mathfrak{C}$. If $\varepsilon > 0$ is such that $\varepsilon + \delta < 1$, by minimality of $\mathfrak{C}$ we can find a $\nu \in {}_s\mathfrak{U}$ and a $\lambda \in \mathfrak{C}$ such that $\nu\langle\lambda\rangle \leqslant \mu + \varepsilon$, i.e., such that $\sup_{t\in X} \lambda(t) \wedge \nu(t, x) \leqslant \mu(x) + \varepsilon$. Taking $B = \lambda^{-1}]\varepsilon + \delta, 1] \in \iota_1(\mathfrak{C})$ and $V = \nu^{-1}]\varepsilon + \delta, 1] \in \iota_u(\mathfrak{U})$, we can find for each $y \in V(B)$ an $x \in B$ such that $(x, y) \in V$, and therefore, $\lambda(x) \wedge \nu(x, y) > \varepsilon + \delta$. It follows that $\varepsilon + \delta < \lambda\langle\nu\rangle(y) \leqslant \mu(y) + \varepsilon$ and so $y \in A$, whence $V(B) \subset A$.

(ii) If $\mathscr{C}$ is a Cauchy filter on $(X, \mathfrak{U})$, by Lemma 3.3 we know that $\omega_1(\mathscr{C}) = \hat{\mathfrak{C}}$, where $\mathfrak{C} = [\{1_F | F \in \mathscr{C}\}]$, and therefore satisfies (HC2), while $c(\omega_1(\mathscr{C})) = 1$ by its definition and thus $c(\hat{\mathfrak{C}}) = c(\mathfrak{C}) = 1$. As to (HC3), let $\nu \in \omega_u(\mathscr{U})$ and $\varepsilon \in I_0$. Then there is a $F_\varepsilon \in \mathscr{C}$ such that

$F_\varepsilon \times F_\varepsilon \subset \nu^{-1}]1-\varepsilon, 1]$, i.e., $1_{F_\varepsilon} \in \mathfrak{C}$ and $1_{F_\varepsilon} \times 1_{F_\varepsilon} - \varepsilon \leqslant \nu$. Since $\mathfrak{C}$ thus satisfies (HC3), $\hat{\mathfrak{C}}$ also does.

If, conversely, $\omega_1(\mathscr{C})$ is a hyper Cauchy prefilter on $(X, \omega_u(\mathscr{U}))$, by part (i), $\mathscr{C} = \iota_1 \circ \omega_1(\mathscr{C})$ is Cauchy on $(X, \iota_u \circ \omega_u(\mathscr{U}) = \mathscr{U})$. If $\mathscr{C}$ is minimal, it suffices to show that $\omega_1(\mathscr{C})_0 \supset \omega_1(\mathscr{C})$. Let $\mu \in \omega_1(\mathscr{C})$ and $\varepsilon \in I_0$. Then from the minimality of $\mathscr{C}$ it follows that there exist $U_\varepsilon \in \mathscr{U}$ and $F_\varepsilon \in \mathscr{C}$ such that

$$\mu^{-1}]1-\varepsilon, 1] \supset U_\varepsilon(F_\varepsilon).$$

If we now put $\nu_\varepsilon = 1_{U_\varepsilon}\langle 1_{F_\varepsilon}\rangle = 1_{U_\varepsilon(F_\varepsilon)} \in \omega_1(\mathscr{C})_0$, it is easily verified that $\nu_\varepsilon - \varepsilon \leqslant \mu$. Since we can find such ν_ε for each $\varepsilon \in I_0$ it follows from (HC2) that $\mu \in \omega_1(\mathscr{C})_0$. If $\omega_1(\mathscr{C})$ is minimal, it follows from (i) that $\mathscr{C} = \iota_1 \circ \omega_1(\mathscr{C})$ is minimal on $(X, \iota_u \circ \omega_u(\mathscr{U}) = \mathscr{U})$.

(iii) The first part follows from (ii) since $\omega_1(\mathscr{C})$ is a hyper Cauchy prefilter on $(X, \omega_u \circ \iota_u(\mathfrak{U}))$, while $\mathfrak{U} \subset \omega_u \circ \iota_u(\mathfrak{U})$. If $\omega_1(\mathscr{C})$ is minimal, then, by (i), $\iota_1 \circ \omega_1(\mathscr{C}) = \mathscr{C}$ is minimal on $(X, \iota_u(\mathfrak{U}))$.

(iv) The first part follows from (i) and $\iota_u \circ \omega_u(\mathscr{U}) = \mathscr{U}$, and similarly the minimality of $\iota_1(\mathfrak{C})$ follows from that of $\mathfrak{C}$. If $\iota_1(\mathfrak{C})$ is minimal, then $\omega_1 \circ \iota_1(\mathfrak{C})$ is minimal on $(X, \omega_u(\mathscr{U}))$ by (ii); but $\mathfrak{C} \subset \omega_1 \circ \iota_1(\mathfrak{C})$, and so $\mathfrak{C} = \omega_1 \circ \iota_1(\mathfrak{C})$.

THEOREM 3.5. *If $(X, \mathscr{U})$ is a uniform space, then $(X, \omega_u(\mathscr{U}))$ is complete if and only if $(X, \mathscr{U})$ is complete.*

Proof. (a) Let $(X, \omega_u(\mathscr{U}))$ be complete and let $\mathfrak{F}$ be a Cauchy filter on $(X, \mathscr{U})$. We consider the prefilter $\mathfrak{F} = [\{1_F | F \in \mathscr{F}\}]$ and first make some remarks.

First, it is obvious that

$$\mathscr{S}_m(\mathfrak{F}) = \{[\{1_G | G \in \mathscr{G}\}] | \mathscr{G} \text{ ultra}, \mathscr{G} \supset \mathscr{F}\};$$

for each such $\mathfrak{G} = [\{1_G | G \in \mathscr{G}\}]$ it is clear that $c(\mathfrak{G}) = 1$ and therefore

$$c^-(\mathfrak{F}) = c(\mathfrak{F}) = 1.$$

Second, if, with the same notations, $1_G \in \mathfrak{G}$ and $(F_\varepsilon)_{\varepsilon \in I_0} \in \mathscr{F}^{I_0}$, then

$$\mu = 1_G \wedge \sup_{\varepsilon \in I_0} (1_{F_\varepsilon} - \varepsilon) \geqslant \sup_{\varepsilon \in I_0} (1_G \wedge 1_{F_\varepsilon} - \varepsilon),$$

while $1_G \wedge 1_{F_\varepsilon} \in \mathfrak{G}$; therefore $\mu \in \hat{\mathfrak{G}}$, and as these generate $\mathfrak{G} \vee \hat{\mathfrak{F}}$, it follows that $\mathfrak{G} \vee \hat{\mathfrak{F}} \subset \hat{\mathfrak{G}}$ and hence

$$c(\mathfrak{G} \vee \hat{\mathfrak{F}}) \geqslant c(\hat{\mathfrak{G}}) = c(\mathfrak{G}) = 1.$$

Third, $\hat{\mathfrak{F}} = \omega_1(\mathscr{F})$ is a hyper Cauchy prefilter on $(X, \omega_u(\mathscr{U}))$. Now

$$\sup_{\mathfrak{C} \in \mathscr{M}(X)} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} c(\mathfrak{C}, \mathfrak{G}) \geqslant \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} c(\hat{\mathfrak{F}}, \mathfrak{G}) = 1 = c(\mathfrak{F}).$$

and so $\mathfrak{F}$ is a Cauchy filter on $(X, \omega_u(\mathscr{U}))$ and therefore convergent.

As for each $\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})$, we can easily verify that

$$\lim \mathfrak{G} = 1_{\lim \mathscr{G}}$$

it now follows that

$$1 = \sup_{x \in X} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} \lim \mathfrak{G}(x) = \sup_{x \in X} \inf_{\substack{\mathscr{G} \supset \mathscr{F} \\ \mathscr{G} \text{ ultra}}} 1_{\lim \mathscr{G}}(x)$$

$$= \sup_{x \in X} 1_{\lim \mathscr{F}}(x),$$

which implies that there exists an $x \in X$ such that $\mathscr{F} \to x$. Thus $(X, \mathscr{U})$ is complete.

(b) Now let $(X, \omega_u(\mathscr{U}))$ be complete. If $\mathfrak{C}$ is a minimal hyper Cauchy prefilter on $(X, \omega_u(\mathscr{U}))$, we know by Lemma 3.4(i) that $\iota_1(\mathfrak{C})$ is a minimal Cauchy filter on $(X, \iota_u \circ \omega_u(\mathscr{U}) = \mathscr{U})$.

Since $(X, \mathscr{U})$ is complete, there exists $x \in X$ such that $\iota_1(\mathfrak{C}) = \mathscr{U}(x)$. Consequently, from [5, Theorem 6.2],

$$\mathfrak{C} \subset \omega_1(\iota_1(\mathfrak{C})) = \omega_1(\mathscr{U}(x)) = \omega_u(\mathscr{U})(x),$$

and since both $\mathfrak{C}$ and $\omega_u(\mathscr{U})(x)$ are minimal, it follows that $\mathfrak{C} = \omega_u(\mathscr{U})(x)$. This proves that

$$\mathscr{M}(X) = \{\omega_u(\mathscr{U})(x) \mid x \in X\}.$$

Now if $\mathfrak{F}$ is a Cauchy prefilter, then

$$\sup_{x \in X} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} \lim \mathfrak{G}(x) = \sup_{x \in X} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} c(\omega_u(\mathscr{U})(x), \mathfrak{G})$$

$$= \sup_{\mathfrak{C} \in \mathscr{M}(X)} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} c(\mathfrak{C}, \mathfrak{G}) = c^{-}(\mathfrak{F}),$$

which proves $(X, \omega_u(\mathscr{U}))$ is complete.

We close this section with a criterion for completeness.

THEOREM 3.5. *If $(X, \mathfrak{U})$ is a fuzzy uniform space, then it is complete if and only if each prefilter $\mathfrak{F}$ fulfilling*

$$\sup_{\mathfrak{C} \in \mathscr{M}(X)} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} c(\mathfrak{C}, \mathfrak{G}) = c(\mathfrak{F}) \tag{*}$$

is convergent.

Proof. If $(X, \mathfrak{U})$ is complete and $\mathfrak{F}$ fulfills (*), then remark that from

$$c(\mathfrak{F}) = \sup_{\mathfrak{C} \in \mathscr{M}(X)} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} c(\mathfrak{C}, \mathfrak{G}) \leqslant \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} c(\mathfrak{G}) = c^-(\mathfrak{F})$$

and Proposition 1.1(iv) it follows that $c(\mathfrak{F}) = c^-(\mathfrak{F})$. Thus $\mathfrak{F}$ is Cauchy and consequently it is convergent.

Conversely, let $\mathfrak{F}$ be Cauchy and $c^-(\mathfrak{F}) > 0$. In [8] it was shown that for the prefilter

$$\mathfrak{F}^* = [\{c^-(\mathfrak{F})\, 1_F \mid F \in \iota_0(\mathfrak{F})\}]$$

we have

(i) $\lim \mathfrak{F} = \lim \mathfrak{F}^*$,

(ii) $c(\mathfrak{F}^*) = c^-(\mathfrak{F}^*) = c^-(\mathfrak{F})$,

(iii) $\iota_0(\mathfrak{F}) = \iota_0(\mathfrak{F}^*)$.

Now let $\varepsilon \in I_0$ and $\mathscr{U} \supset \iota_0(\mathfrak{F})$ be ultra. Then, since $\mathfrak{F}$ is Cauchy, we can find $\mathfrak{C}_\varepsilon \in \mathscr{M}(X)$ such that $c(\mathfrak{C}_\varepsilon, (\mathfrak{F}, \mathscr{U})) \geqslant c^-(\mathfrak{F}) - \varepsilon$. Consequently,

$$\begin{aligned}
c(\mathfrak{C}_\varepsilon, (\mathfrak{F}^*, \mathscr{U})) &= \inf_{\nu \in \mathfrak{C}_\varepsilon} \inf_{F \in \iota_0(\mathfrak{F})} \inf_{U \in \mathscr{U}} \sup_{y \in X} c^-(\mathfrak{F})\, 1_F \wedge 1_U \wedge \nu(y) \\
&= \inf_{\nu \in \mathfrak{C}_\varepsilon} \inf_{U \in \mathscr{U}} \sup_{y \in U} c^-(\mathfrak{F}) \wedge \nu(y) \\
&\geqslant c^-(\mathfrak{F}) \wedge \left(\inf_{\nu \in \mathfrak{C}_\varepsilon} \inf_{U \in \mathscr{U}} \inf_{\mu \in \mathfrak{F}} \sup_{y \in U} \nu \wedge \mu(y) \right) \\
&\geqslant c^-(\mathfrak{F}) \wedge (c^-(\mathfrak{F}) - \varepsilon) = c^-(\mathfrak{F}) - \varepsilon = c(\mathfrak{F}^*) - \varepsilon.
\end{aligned}$$

This proves that $\mathfrak{F}^*$ fulfills condition $(*)$ and consequently is convergent. Then it follows that

$$\sup_{x \in X} \lim \mathfrak{F}(x) = \sup_{x \in X} \lim \mathfrak{F}^*(x) = c^-(\mathfrak{F}^*) = c^-(\mathfrak{F}),$$

which proves that also $\mathfrak{F}$ is convergent. Consequently, $(X, \mathfrak{U})$ is complete.

DEFINITION 3.3. A prefilter $\mathfrak{F}$ in a fuzzy uniform space is called *upper Cauchy* if and only if

$$\sup_{\mathfrak{C} \in \mathscr{M}(X)} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} c(\mathfrak{C}, \mathfrak{G}) = c(\mathfrak{F}).$$

Theorem 3.5 then says that a fuzzy uniform space is already complete if each upper Cauchy prefilter is convergent. It also shows that replacing c^- by c in both the definitions of convergent and of Cauchy prefilter does not affect the notion of completeness. One may well ask then why c^- and not c? The fundamental reasons providing the answer to this question are not within the scope of the present paper but can be found in [5, 8].

We can, however, add that with c instead of c^-, Proposition 2.7 would no longer be true. An example of this fact is given by taking $X = I$ and $\mathfrak{U} = \{\alpha\}^{\sim}$, where $\alpha = \frac{1}{2} \vee 1_{D(X)}$, $D(X)$ being the diagonal of X. Here no other prefilters than the $\dot{1}_x$ $(x \in X)$ are at the same time upper Cauchy and hyper Cauchy, while nevertheless neither of these two classes of prefilters reduces to $\{\dot{1}_x | x \in X\}$.

4. COMPACTNESS

PROPOSITION 4.1. *If $(X, \mathfrak{U})$ is a fuzzy uniform space, then each fuzzy entourage is a fuzzy neighborhood of the diagonal $D(X)$ of $X \times X$.*

Proof. Let $\nu \in \mathfrak{U}$ and $\varepsilon \in I_0$. Choose $\xi \in {}_s\mathfrak{U}$ such that $\xi \circ \xi - \varepsilon \leqslant \nu$; then $\xi\langle x \rangle \times \xi\langle x \rangle \in \mathfrak{U} \times \mathfrak{U}(x, x)$ for all $x \in X$ [6] and thus $\sup_{x \in X} \xi\langle x \rangle \times \xi\langle x \rangle$ is a fuzzy neighborhood of $D(X)$. Since $\sup_{x \in X} \xi\langle x \rangle \times \xi\langle x \rangle - \varepsilon \leqslant \nu$ this proves that ν, too, is a fuzzy neighborhood of $D(X)$.

THEOREM 4.1. *If $(X, \mathfrak{U})$ is a compact fuzzy uniform space, then $\mathfrak{U}$ consists exactly of all fuzzy neighborhoods of the diagonal $D(X)$ of $X \times X$, and thus no other fuzzy uniformity on X induces the same fuzzy topology on X as does $\mathfrak{U}$.*

Proof. Let $(X, \mathfrak{U})$ be compact and let $\mu = \sup_{x \in X} \nu_x\langle x \rangle \times \nu_x\langle x \rangle$, where $\nu_x \in {}_s\mathfrak{U}$ for all $x \in X$. Clearly, μ is a fuzzy neighborhood of $D(X)$. For any $x \in X$ and any $\varepsilon \in I_0$, choose $\xi_x \in {}_s\mathfrak{U}$ such that $\xi_x \circ \xi_x - \varepsilon/2 \leqslant \nu_x$. From the compactness of $(X, \mathfrak{U})$ it follows that there exists a finite subset $Y \subset X$ such that $\sup_{x \in Y} \xi_x\langle x \rangle \geqslant 1 - \varepsilon/2$. Let $\xi = \inf_{x \in Y} \xi_x \in \mathfrak{U}$; then for all $(u, v) \in X \times X$ we have

$$\begin{aligned}\mu(u,v) &= \sup_{x\in X} (\nu_x\langle x\rangle(u) \wedge \nu_x\langle x\rangle(v))\\
&\geqslant \sup_{x\in X} ((\xi_x\circ\xi_x)\langle x\rangle(u) \wedge (\xi_x\circ\xi_x)\langle x\rangle(v)) - \varepsilon/2\\
&\geqslant \sup_{x\in X} \left(\left(\sup_{t\in X} \xi_x(u,t)\wedge\xi_x(t,x)\right)\wedge\left(\sup_{s\in X}\xi_x(x,s)\wedge\xi_x(s,v)\right)\right) - \varepsilon/2\\
&\geqslant \sup_{x\in X}\sup_{t\in X} (\xi_x(u,t)\wedge\xi_x(t,x)\wedge\xi_x(t,v)) - \varepsilon/2\\
&\geqslant \sup_{x\in Y}\sup_{t\in X} (\xi(u,t)\wedge\xi_x(t,x)\wedge\xi(t,v)) - \varepsilon/2\\
&\geqslant \sup_{t\in X}\left(\left(\sup_{x\in Y}\xi_x(t,x)\right)\wedge\xi(u,t)\wedge\xi(t,v)\right) - \varepsilon/2\\
&\geqslant \sup_{t\in X} ((1-\varepsilon/2)\wedge\xi(u,t)\wedge\xi(t,v)) - \varepsilon/2\\
&\geqslant \xi\circ\xi(u,v) - \varepsilon.\end{aligned}$$

Since we can find such ξ for each $\varepsilon \in I_0$, and since $\xi\circ\xi \in \mathfrak{U}$, this proves $\mu\in\mathfrak{U}$.

The uniqueness of $\mathfrak{U}$ follows immediately from [5, Theorem 4.1] (uniqueness of a fuzzy neighborhoodspace structure associated with a given fuzzy closure operator).

Making use of the definition of initial fuzzy uniformities, cf. [6], we have trivially

PROPOSITION 4.2. *If (X, Δ) is a fuzzy uniformizable space, there exists a largest fuzzy uniformity between all fuzzy uniformities associated with Δ.*

Writing as usual $t(\mathfrak{U})$ for the fuzzy topology associated with the fuzzy uniformity $\mathfrak{U}$, we can now formulate

DEFINITION 4.1. A fuzzy uniform space $(X, \mathfrak{U})$ is called *fine* if $\mathfrak{U}$ is the largest fuzzy uniformity associated with the fuzzy topology $t(\mathfrak{U})$.

Just as in ordinary topology, we then have immediately

PROPOSITION 4.3. *The fuzzy uniform space $(X, \mathfrak{U})$ is fine if and only if for each fuzzy uniform space $(X', \mathfrak{U}')$ and each continuous $f: X, t(\mathfrak{U}) \to X'$, $t(\mathfrak{U}')$, the mapping $f: X, \mathfrak{U} \to X', \mathfrak{U}'$ is uniformly continuous.*

Proof. The *only if* part is trivial. For the *if* part, it is sufficient to consider the fuzzy uniformity $\mathfrak{U}_0 = (f \times f)^{-1}(\mathfrak{U}')$, $t(\mathfrak{U}_0) = f^{-1}(t(\mathfrak{U}'))$, and to observe that $t(\sup(\mathfrak{U}, \mathfrak{U}_0)) = \sup(t(\mathfrak{U}), t(\mathfrak{U}_0)) = t(\mathfrak{U})$.

In view of Proposition 4.3 and Theorem 4.1. we get

THEOREM 4.2. *If $(X, \mathfrak{U})$ is a compact fuzzy uniform space, $(X', \mathfrak{U}')$ is a fuzzy uniform space, and $f : X, t(\mathfrak{U}) \to X', t(\mathfrak{U}')$ a continuous map, then $f : X, \mathfrak{U} \to X', \mathfrak{U}'$ is uniformly continuous.*

That Definition 4.1 is a good extension follows from

PROPOSITION 4.4. *If $(X, \mathscr{U})$ is a uniform space, then $(X, \omega_u(\mathscr{U}))$ is fine if and only if $(X, \mathscr{U})$ is fine.*

Proof. We use the notations of [6, Theorem 3.1]. It follows from this theorem that $\iota(t(\omega_u(\mathscr{U}))) = T(\iota_u(\omega_u(\mathscr{U}))) = T(\mathscr{U})$, $\iota(t(\mathfrak{U}')) = T(\iota_u(\mathfrak{U}'))$, $\mathfrak{U}' \subset \omega_u(\iota_u(\mathfrak{U}'))$ (if $(X, \mathfrak{U}')$ is a fuzzy uniform space). Then, if $(X, \mathscr{U})$ is fine, by [6, Theorem 3.3; 3, Proposition 3.1 and Corollary 3.2] (note, however, that the words *fuzzy continuous* in [3] are replaced by *continuous*), we have successively, $f : X, t(\omega_u(\mathscr{U})) \to X', t(\mathfrak{U}')$ continuous $\Rightarrow f : \iota(t(\omega_u(\mathscr{U}))) \to X', \iota(t(\mathfrak{U}'))$ continuous $\Rightarrow f : X, T(\mathscr{U}) \to X', T(\iota_u(\mathfrak{U}'))$ continuous $\Rightarrow$ $f : X, \mathscr{U} \to X'$, $\iota_u(\mathfrak{U}')$ uniformly continuous $\Rightarrow f : X, \omega_u(\mathscr{U}) \to X', \omega_u(\iota_u(\mathfrak{U}'))$ uniformly continuous $\Rightarrow f : X, \omega_u(\mathscr{U}) \to X', \mathfrak{U}'$ uniformly continuous.

The converse is shown analogously.

THEOREM 4.3. *If $(X, \mathfrak{U})$ is a fuzzy uniform space and $Y \subset X$ is a compact subspace, then the family of all fuzzy neighborhoods of the type $\nu\langle 1_Y\rangle$. where $\nu \in \mathfrak{U}$, forms a basis for the fuzzy neighborhoodsystem of Y.*

Proof. For each $y \in Y$, choose $\nu_y \in \mathfrak{U}$. Then $\sup_{y \in Y} \nu_y\langle y\rangle$ is a fuzzy neighborhood of Y. Let $\varepsilon \in I_0$ and choose, for each $y \in Y$, a $\xi_y \in {}_s\mathfrak{U}$ such that $\xi_y \circ \xi_y - \varepsilon/2 \leqslant \nu_y$. From the compactness of Y it follows that there exist $y_1, \dots, y_n \in Y$ such that

$$\sup_{i=1}^{n} \xi_{y_i}\langle y_i\rangle \geqslant 1 - \varepsilon/2 \text{ on } Y.$$

Set $\xi = \inf_{i=1}^{n} \xi_{y_i} \in {}_s\mathfrak{U}$. Then for any $x \in X$ we have

$$\xi\langle 1_Y\rangle(x) - \varepsilon = \sup_{t \in X} 1_Y(t) \wedge \xi(t, x) - \varepsilon$$

$$\leqslant \sup_{t \in Y} \xi(t, x) \wedge \left(\sup_{i=1}^{n} \xi_{y_i}\langle y_i\rangle(t) + \varepsilon/2\right) - \varepsilon$$

$$\leqslant \sup_{t\in Y} \sup_{i=1}^{n} \xi_{y_i}(y_i, t) \wedge \xi(t, x) - \varepsilon/2$$

$$\leqslant \sup_{i=1}^{n} \xi_{y_i} \circ \xi_{y_i}(y_i, x) - \varepsilon/2$$

$$\leqslant \sup_{i=1}^{n} \nu_{y_i}\langle y_i\rangle(x) \leqslant \sup_{y\in Y} \nu_y\langle y\rangle(x),$$

which proves the theorem.

THEOREM 4.4. *A fuzzy uniform space* $(X, \mathfrak{U})$ *is compact if and only if it is precompact and complete.*

Proof. The *if* part follows at once from [4, Theorem 5.1] and from Theorem 3.1(iv). To show the *only if* part, let $(X, \mathfrak{U})$ be compact. Precompactness again follows at once from Theorem 1.2 and Definition 3.1. To prove that $(X, \mathfrak{U})$ is complete, let $\mathfrak{F}$ be a Cauchy prefilter; then for all $\varepsilon \in I_0$ there exists $\mathfrak{C} \in \mathscr{M}(X)$ such that

$$\inf_{\mathfrak{G}\in \mathscr{P}_m(\mathfrak{F})} c(\mathfrak{C}, \mathfrak{G}) > c^-(\mathfrak{F}) - \varepsilon/3. \tag{1}$$

Since X is compact and $c(\mathfrak{C}) = 1$, we can find $x \in X$ such that

$$c(\mathfrak{C}, \mathfrak{U}(x)) = \text{adh } \mathfrak{C}(x) > 1 - \varepsilon/3. \tag{2}$$

If now $\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})$, $\mu \in \mathfrak{G}$, $\nu \in \mathfrak{U}$, choose $\theta \in {}_s\mathfrak{U}$ such that $\theta \circ \theta - \varepsilon/3 \leqslant \nu$, and also $\xi \in \mathfrak{C}$, $\mathfrak{C}$ being hyper Cauchy, such that $\xi \times \xi - \varepsilon/3 \leqslant \theta$. From (1) it follows that there exists a $y_0 \in X$ for which

$$\mu(y_0) \wedge \xi(y_0) > c^-(\mathfrak{F}) - \varepsilon/3,$$

while from (2) it follows that there exists a $y_1 \in X$ such that

$$\xi(y_1) \wedge \theta(y_1, x) > 1 - \varepsilon/3.$$

Consequently

$$\begin{aligned}
\mu(y_0) \wedge \nu(x, y_0) &\geqslant \mu(y_0) \wedge (\theta \circ \theta(x, y_0) - \varepsilon/3) \\
&\geqslant \mu(y_0) \wedge \theta(y_0, y_1) \wedge \theta(y_1, x) - \varepsilon/3 \\
&\geqslant \mu(y_0) \wedge (\xi(y_0) \wedge \xi(y_1) - \varepsilon/3) \wedge \theta(y_1, x) - \varepsilon/3 \\
&\geqslant \mu(y_0) \wedge \xi(y_0) \wedge \xi(y_1) \wedge \theta(y_1, x) - 2\varepsilon/3 \\
&\geqslant (c^-(\mathfrak{F}) - \varepsilon/3) \wedge (1 - \varepsilon/3) - 2\varepsilon/3 \geqslant c^-(\mathfrak{F}) - \varepsilon.
\end{aligned}$$

Since this holds for all $\mu \in \mathfrak{G}$ and $\nu \in \mathfrak{U}$, we have $\inf_{\mu \in \mathfrak{G}} \inf_{\nu \in \mathfrak{U}} \sup_{y \in X} \mu(y) \wedge \nu(x, y) \geqslant c^-(\mathfrak{F}) - \varepsilon$, i.e. $c(\mathfrak{G}, \mathfrak{U}(x)) \geqslant c^-(\mathfrak{F}) - \varepsilon$.

All together this means that for each $\varepsilon \in I_0$ we have $x \in X$ such that $c(\mathfrak{G}, \mathfrak{U}(x)) \geqslant c^-(\mathfrak{F}) - \varepsilon$ for all $\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})$, and from this it follows that $\sup_{x \in X} \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} c(\mathfrak{G}, \mathfrak{U}(x)) = c^-(\mathfrak{F})$, which proves the theorem.

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